

Jost Bürgi's methods of calculating sines, and possible transmission from India

Journal Article**Author(s):**

Wagner, Roy ; Hunziker, Samuel

Publication date:

2019-05-06

Permanent link:

<https://doi.org/10.3929/ethz-b-000313142>

Rights / license:

[In Copyright - Non-Commercial Use Permitted](#)

Originally published in:

Archive for History of Exact Sciences 73(3), <https://doi.org/10.1007/s00407-019-00225-z>



Jost Bürgi's methods of calculating sines, and possible transmission from India

Roy Wagner¹ · Samuel Hunziker¹

Received: 20 February 2019 / Published online: 14 March 2019
© Springer-Verlag GmbH Germany, part of Springer Nature 2019

Abstract

A few years ago, a manuscript by Jost Bürgi (1552–1632) was brought to scholarly attention, which included an ingenious sine calculation method. The purpose of this paper is to discuss two aspects of this manuscript. First, we wish to improve the current understanding of Bürgi's method of sine calculation, especially with respect to the calculation of sines at a resolution of 1 min. Second, we wish to suggest a possible transfer of knowledge between India's Kerala School of mathematical astronomy and Bürgi. The evidence for the latter seems to be stronger than the evidence for other available case studies, but still revolves mainly around analogies, and can therefore not be considered as conclusive proof of transmission. We also append a translation of the relevant chapter of Bürgi's treatise.

1 Introduction

Our main protagonist, the instrument maker, mathematician, astronomer, and metallurgist Jost Bürgi was born 1552 in Lichtensteig, Switzerland. Nothing is known about Bürgi's education and early life, but as he was not proficient in Latin, it seems he did not get a university education (Staudacher 2018, 53ff; for a summary English biography see Clark 2015, 1–12). In 1579, Bürgi resurfaces in surviving documents when he becomes the court clockmaker of the Landgrave of Hesse-Kassel, William IV. The latter is known for his interest in astronomical research and as the creator of the first permanent observatory in Europe (von Mackensen 1979; Staudacher

Communicated by Noel Swerdlow.

✉ Samuel Hunziker
samuel.hunziker@gess.ethz.ch

Roy Wagner
roy.wagner@gess.ethz.ch

¹ Chair of History and Philosophy of Mathematical Sciences, ETH Zürich, RZ J 6, Clausiusstrasse 59, 8092 Zurich, Switzerland

2018, 117). While in Kassel, Bürgi built many of his most innovative and famous instruments.

Throughout his career, Bürgi constructed armillary spheres, celestial globes, and a variety of astronomical instruments, including a precise and mobile sextant that was used by Brahe (Šíma 1993). Some of his clocks were probably the most accurate of his time (Staudacher 2018, 123). Bürgi also participated in the observations conducted by the Landgrave and the court mathematician and astronomer Christoph Rothmann. After Rothmann's departure, Bürgi became acting court mathematician and astronomer for 7 years (143), until Johannes Hartmann took over in 1597 (251).

Bürgi wrote his most important mathematical works during his time in Kassel. He presented his *Artificium*, the method for calculating sines that will be discussed below, in his unpublished *Fundamentum Astronomiae* (Bürgi 1592; rediscovered by Menso Folkerts; published by Launert 2015). His lost *Canon Sinuum*, which contained eight-digit sine tables in steps of two arc-seconds, was praised by Kepler (Staudacher 2018, 196). As an introduction to his *Canon Sinuum*, Bürgi wrote the *Coss*, a treatise on Cossic algebra with applications to geometry that provided methods for astronomy and spherical trigonometry; he finished it around 1603 (published by List and Bialas 1973). Furthermore, Bürgi invented a version of logarithms and created tables independently of John Napier. These tables most likely preceded those of Napier, but were only published in 1620 (Staudacher 2018, 113). As we can see, Bürgi was hesitant in publishing. In fact, he actively forbade people familiar with his work from sharing it (228).¹

In 1592, Bürgi travelled to Prague for an audience with the emperor of the Holy Roman empire, Rudolf II. He gave him, among other things, his manuscript *Fundamentum Astronomiae*. After multiple visits, Bürgi permanently moved to Prague in 1604, where he stayed until 1630. In Prague, Bürgi got to know imperial court mathematician Brahe, who was there from 1597 until his death in 1601, and his successor Kepler, who was in Prague between 1600 and 1612 and with whom Bürgi worked closely. There is evidence that Bürgi substantially contributed to Kepler's work through his instruments, observation data, and mathematical innovations (Gaulke 2007; Staudacher 2018, ch. 13).

The other scene of this paper, Kerala, was also a boisterous hub with strong commerce relations to the Arabic world and, to a lesser extent, with European merchants. Along with the indigenous communities, it was home to substantial Muslim, Syriac-Christian, and Jewish communities. Since Vasco de Gama's expedition in 1498, it also had direct contact with European ships and military powers.

The Kerala School of mathematical astronomy is the contemporary name associated with a group of Indian scholars whose work survives in Sanskrit and Malayalam manuscripts. It originated with fourteenth-century scholar Mādhava, and continued through teacher-student transmission at least until the seventeenth centuries (since the precise "boundaries" of this school are not well defined, the scope and dates may vary in different historiographies). Two of the main treatises of this school

¹ Staudacher speculates that Bürgi might have been dyslexic, felt inadequate among people from a higher class, and feared Brahe and the ways he dealt with other scholars (see 2018, 227ff).

are available in high-quality modern editions with English translations (Sarma et al. 2009; Ramasubramanian and Sriram 2010).

It is rather safe to say that the mathematical culture in Kerala was the most advanced of its time worldwide. This does not mean that all knowledge available to mathematicians in Asia, Africa, and Europe was also available to the Kerala school (for example, they had no discussions of conic sections), but the latter's methods and innovations made up for the difference. Among other things, the Kerala school pre-figured European calculus with power-series-like expansion for trigonometric functions and the value of π and anticipated Brahe's astronomical models (for surveys of Kerala school mathematics, see Sarma 1972; Joseph 2009a; Plofker 2009, ch. 7; Divakaran 2018, chs. 9–13).

2 Bürgi's *Artificium*

Bürgi's *Artificium* is a highly ingenious method for producing sine tables at fixed intervals from 0° to 90° . Note that Bürgi does not calculate the normalized sine values for a circle with radius 1 (as we do today), but the *Rsines*, that is, the lengths of half-chords in a circle of radius R . This requires attention because Bürgi's procedure involves a different value of R in each iteration (column) of the calculation. The method goes as follows:

- Create a table with as many columns as you want (a larger number of columns will yield higher precision). The cells in the odd columns correspond to the evenly spaced angles at which sines are calculated (e.g., $0^\circ, 10^\circ, 20^\circ, \dots, 90^\circ$). The cells in the even columns are aligned between the cells of the odd columns (see Fig. 1).
- On the right-most column, insert some initial values (these may be thought of as initial estimates for the *Rsines*, but they do not have to be good estimates, and Bürgi does not call them estimates). The radius R , being $R \sin 90^\circ$, is the last value in the column.
- To fill in an even column, divide the value in the bottom cell (that is, the radius) of the previous odd column in half, and put it at the bottom cell of the even column. To fill in the cell directly above it, sum the number below and the number between the two cells in the odd column immediately to the right. Continue until you reach the top of the column.
- To fill in an odd column, place zero at the top cell (which equals $R \sin 0^\circ$). To fill in the cell directly below it, sum the number above and the number between the two cells in the even column immediately to the right. Continue until you reach the bottom of the column.

The odd columns then provide improving approximations of *Rsine* values, where the radius R is the value at the bottom of the column. The even columns approximate the *Rsine* differences between the adjacent angles (or, as it turns out, R' cosines of the

	sinus 5	sinus 4	sinus 3	sinus 2	sinus 1
0	0.0.0.0	0.0.0.0	0.0.0.0	0.0.0	0
10	10.20.51.0	0.18.51.52	0.34.24	1.3	2
20	20.22.50.8	0.37.9.20	1.7.45	2.4	4
30	29.47.39.56	0.54.19.3	1.39.2	3.1	6
40	38.18.10.41	1.9.49.44	2.7.18	3.52	7
50	45.38.51.42	1.23.13.7	2.31.42	4.36	8
60	51.36.19.36	1.34.4.48	2.51.30	5.12	9
70	55.59.42.42	1.42.4.59	3.6.6	5.39	10
80	58.41.0.49	1.46.59.4	3.15.3	5.56	11
90	59.35.19.52	1.48.38.6	3.18.4	6.2	12

Fig. 1 Bürigi’s example for the construction of a sine table in sexagesimal notation (Bürigi 1592, 36r)

intermediate angles, because $R \sin(j + 1)\alpha - R \sin j\alpha$ is proportional to $\cos\left(j + \frac{1}{2}\right)\alpha$. The procedure should be terminated when the ratios of the Rsines in subsequent columns no longer change with respect to the desired level of accuracy.

Bürigi does not prove his procedure. He even states (see the appended translation) that an arithmetic construction such as this one can only be tested or verified, not proved (but we should be careful when projecting our modern interpretations on Bürigi’s terms here). Bürigi’s tests verify that:

- (1) the value of the Rsine of 30° is half the radius;
- (2) the sum of squares of the Rsine of an arc and of the Rsine of its complement is the square of the radius;
- (3) the difference between the Rsines of $(60 + x)^\circ$ and $(60 - x)^\circ$ equals $R \sin x^\circ$;
- (4) the ratios between the same Rsines in different odd columns (or Rsine differences in different even columns) are approximately the same and become more stable as we proceed to the left hand columns;
- (5) the Rsines (or their ratios) correspond to those derived by analyzing the sides of inscribed polygons (or their ratios), as in the sixth chapter of Bürigi’s treatise (Lauert 2015, Ch. 6).

3 How did Bürigi discover his *Artificium*?

It is plausible that the basic observation underlying the method is that the second differences of sines are proportional to the sines themselves (although Ullrich suggests a different approach).² To be precise, define $RD_{j,\alpha}^2$ to be the second difference of Rsine values at intervals α starting at 0° and ending at 90° , namely

² Ullrich (2016, forthcoming) suggests that Bürigi’s method could have arisen from an iterative application of *prosthaphaeresis* rather than from the proportionality of sines and their second differences.

$$R\Delta_{j,\alpha}^2 = (R \sin j\alpha - R \sin (j - 1)\alpha) - (R \sin (j + 1)\alpha - R \sin j\alpha).$$

Now, if one starts with arbitrary values and then calculates their differences, then differences of differences, etc. (as when one reads Bürgi's table above from left to right), the results will typically diverge away from the values of *Rsines*. For example, if one starts with a linear estimate on the left and proceeds to the right, one simply gets a column of zeros as the next *Rsine* approximation. However, if one reverses the procedure, starting with arbitrary values on the right and taking their partial sums as one proceeds to the left, one will typically converge to *Rsine* values (for a modern analysis see Folkerts et al. 2016; Waldvogel 2016; Nicollier 2018).

How did Bürgi come up with the basic observation of proportionality between second differences and *Rsines*? One possibility is suggested by Roegel (2015) and is empirical in nature. Differences were sometime recorded in *Rsine* tables, and one could realize empirically that second differences are proportional to the original *R sines*.

Another suggestion is found in Folkerts et al. (2016, 138, 145), Ullrich (2016), and Waldvogel (2016, 92). The proportionality of sines and their second differences can be easily derived from the so-called *prosthaphaeresis* identity, which Bürgi knew and used in his work (Lauert 2015, Ch. 3):

$$\sin \alpha \cdot \sin \beta = \frac{1}{2}(\sin (90^\circ - \alpha + \beta) - \sin (90^\circ - \alpha - \beta)),$$

Once the basic observation about second differences is derived, it can lead to the discovery of Bürgi's iterative method (see the end of this section).

We suggest here a third possibility. The following formula is proved in treatises of the Kerala School of mathematical astronomy:

$$R\Delta_{j,\alpha}^2 = \left(\frac{Rcrd\alpha}{R}\right)^2 \cdot R \sin j\alpha, \tag{1}$$

where *Rcrdα* is the full chord of the angle α in a circle of radius *R*. Specifically, this formula can be found in early 16th century treatises such as the *Tantrasaṅgraha* (Subramanian and Sriram 2010, 62) and the *Yuktibhāṣa* (Sarma et al. 2009, 224), where it is attributed to 14th century mathematician Mādhava. The same mathematician is also credited there with formulas for the *Rsine* of a sum of angles, which are, in a sense, the inverse of the *prosthaphaeresis* identity.

An identity corresponding to formula (1) is indeed stated in Bürgi's text, using $2(1 - \cos \alpha)$ instead of the chord term (which is indeed equivalent). However, as is clear from a comparison of our reconstruction (in the section on the calculation of the second difference below) and our translation of steps III–IV of the 11th chapter, Bürgi's allusion to this identity is quite vague. Moreover, its justification in Bürgi's text already assumes the proportionality of the second difference and the sine.

The identity (1) (or even just the proportionality of sines and their second differences) is indeed sufficient to partially justify the use of the *Artificium* (bracketing the question of convergence), but not necessarily to discover it. If the source of Bürgi's discovery is, as Roegel suggests, an empirical observation of sine tables, then the

road to the *Artificium* is more or less paved. But if Bürigi first derived independently or learned from others the identity (1) or the proportionality of sines and their differences, then the derivation of the *Artificium* still needs to be explained.

We suggest to think of the *Artificium* as a fixed-point iteration. A fixed-point iteration for solving an equation of the form $f(x) = x$ is a sequence $f(x_n) = x_{n+1}$, which converges to a solution. In our case, we have a collection of equations of the template (1) with integers $0 < j < \frac{1}{\alpha}$. A collection of approximations for $R \sin j\alpha$, for all relevant j 's—that is, an odd column in Bürigi's table—plays the role of x_n . The next odd column is the result of applying the second difference, which is indeed the left hand side of Eq. (1). Since the scale is arbitrary, we can ignore the proportion factor on the right hand side of (1). The next odd column is therefore the next fixed-point iterate, x_{n+1} .

Fixed-point iterations were indeed known in India, the Arabic world,³ and Europe (Plofker 2011; see Wagner 2015 for a case study), but Bürigi's method, according to the above suggestion, would be a highly elaborate use of this principle, which goes quite far beyond earlier uses known to us. Therefore, even if Bürigi obtained identity (1) from the outside, the invention of the *Artificium* seems to be his own achievement.

3.1 The bottom term of even columns in Bürigi's *Artificium*

In order to complete the method, one has to figure out an initial condition: the bottom value of even columns. This is not a partial sum of previous values, but half the bottom value in the previous odd column. Roegel (2015) explains this empirically as well, by looking at tables of differences and differences of differences constructed from good approximations of *R*sine values (going from left to right). Indeed, while a second difference column will not actually contain the radius (as the bottom cell is not a difference of cells in the previous column), the radius can be approximately reconstructed from the available *R*sine values in the column.

While this is plausible, we can suggest non-empirical reconstructions as well. Bürigi may have realized (by reasoning with *prosthaphaeresis*) that the even columns list the *R*cosines of the intermediate angles (or, as he would put it, the *R*sines of their complements with respect to a right angle). So the two bottom cells of an even column should contain (approximations of) the values $R' \sin \frac{\alpha}{2}$ and $R' \sin \frac{3\alpha}{2}$ for some radius R' . Their difference should be the content of the cell above the bottom cell of the given odd column, namely approximately R . We get the estimate $R' \left(\sin \frac{3\alpha}{2} - \sin \frac{\alpha}{2} \right) \approx R$. For a small α , this is approximated by $R' \left(3 \sin \frac{\alpha}{2} - \sin \frac{\alpha}{2} \right) = 2R' \sin \frac{\alpha}{2} \approx R$. Therefore, the bottom number of the even column, $R' \sin \frac{\alpha}{2}$, should be taken as $\frac{R}{2}$. Alternatively, one can use *prosthaphaeresis* to obtain the same result by means of a precise calculation.

³ The fact that Bürigi's procedure goes from right to left may suggest an Arabic source, but we cannot find any further support for this hypothesis.

3.2 The calculation of the second difference

Bürgi's *Artificium* is ingenious, but is also, as he explicitly admits, too tedious for a resolution as fine as one arc minute. To calculate sines at this resolution, Bürgi uses another method. As we will see below, this method depends on estimating the value of $R\Delta_{1,1'}^2 = (R \sin 1' - R \sin 0) - (R \sin 2' - R \sin 1')$, namely the second difference of sines at the first arc minute. This means that this difference has to be calculated *before* calculating the values of sines at 1' resolution. We did not find a reconstruction of this achievement in the literature, so we offer the following, based on our translation of the text below (but note that this translation requires some reconstructions that may be questioned).

Bürgi's first step is to estimate $R \sin 1'$ simply as $\frac{R \sin 1^\circ}{60}$, acknowledging that this is an underestimate. Then, the R sine of $89^\circ 59'$ can be estimated as $\sqrt{R^2 - \left(\frac{R \sin 1^\circ}{60}\right)^2}$, which Bürgi affirms as an overestimate. These are steps I and II in Bürgi's text.

The next thing to figure out (step III in Bürgi's text) is the ratio between the radius of one odd column and that of the next odd column. We do have this figure for the already calculated *Artificium* with a 1° resolution, but not for an *Artificium* with a 1' resolution, which is not yet constructed. However, if we had the latter *Artificium*, the difference between the bottom terms of an odd column would equal (approximately) the bottom term of the preceding even column, which is half the bottom term of the preceding odd column. Denoting the radius of the former odd column \bar{R} and that of the latter odd column R , we get the equation: $\bar{R} - \bar{R} \sin 89^\circ 59' = \frac{R}{2}$, or equivalently, $R : \bar{R} :: 2(1 - \sin 89^\circ 59') : 1$.

Finally, we need to figure out the second difference of the first minute, $R\Delta_{1,1'}^2$. This is done in Bürgi's step IV, if we managed to reconstruct it correctly. By the proportionality of sines and the second differences, the first difference of differences in the odd column with radius \bar{R} , namely $\bar{R}\Delta_{1,1'}^2$, is simply the value of $R \sin 1'$. But if we want to figure out $R\Delta_{1,1'}^2$ for the radius R , we need to rescale this value by the ratio $R \sin 1^\circ : \bar{R} \sin 1^\circ$, which equals $R : \bar{R}$. So, based on step III, $R\Delta_{1,1'}^2$ is $2(1 - \sin 89^\circ 59') \cdot R \sin 1'$ [which is actually a variation of identity (1) above]. Using the above approximations (from Bürgi's steps I and II), we get that $R\Delta_{1,1'}^2$ is approximated by $2\left(1 - \sqrt{1 - \left(\frac{\sin 1^\circ}{60}\right)^2}\right) \cdot R \frac{\sin 1^\circ}{60}$.

Now, starting from $\sin 1^\circ$ as calculated by Bürgi's *Artificium* (that is, correct up to 5 sexagesimal places), one obtains an approximation of $\Delta_{1,1'}^2$, valued at $2.4610 \dots \cdot 10^{-11}$ (the correct value is $2.4614 \dots \cdot 10^{-11}$). Using this value to calculate the R sines of minutes of the first degree by means of Bürgi's method as described below does indeed provide correct values up to 5 sexagesimal places.

3.3 The calculation of the sine of 1 min

Given approximate values for $R \sin 1^\circ$ and for $R\Delta_{1,1'}^2$, Bürgi proposes the approximation $R \sin 1' \approx \frac{R \sin 1^\circ}{60} + \left(\frac{30 \cdot 60}{4} + 2 \frac{(1+2+\dots+29) \cdot \frac{31}{3}}{60}\right) R\Delta_{1,1'}^2$, but does not justify or motivate

this procedure in any way.⁴ Based on later European methods, Roegel (2016) reconstructs this as following from the formula:

$$\begin{aligned} \sin 1^\circ &= \sum_{i=0}^{59} \Delta_{i,1'} = 60\Delta_{1,1'} + \left(\Delta_{1,1'}^2 + \left(\Delta_{1,1'}^2 + \Delta_{2,1'}^2 \right) + \dots + \left(\Delta_{1,1'}^2 + \Delta_{2,1'}^2 + \dots + \Delta_{59,1'}^2 \right) \right) \\ &\approx 60\Delta_{1,1'} + \left(\sum_{i=1}^{59} i \right) \Delta_{1,1'}^2 + \left(\sum_{i=1}^{58} \sum_{j=1}^i j \right) \Delta_{1,1'}^3 \approx 60\Delta_{1,1'} + \left(\sum_{i=1}^{59} i + \sum_{i=1}^{58} \sum_{j=1}^i j \right) \Delta_{1,1'}^2, \end{aligned}$$

where one uses that $\Delta_{1,1'}^2 \approx \Delta_{1,1'}^3$. This reconstruction indeed provides the correct coefficient for $\Delta_{1,1'}^2$, but the separate terms $\frac{59 \cdot 60}{2} + \frac{58 \cdot 59 \cdot 60}{6}$ do not fit Būrgi’s terms, and Būrgi never mentions a third-order difference.

We therefore suggest deriving Būrgi’s calculation from the following formula:

$$\begin{aligned} R \sin n\alpha &= Rn \sin \alpha - \left(\frac{Rcrd\alpha}{R} \right)^2 R(\sin \alpha + (\sin \alpha + \sin 2\alpha) \\ &\quad + \dots + (\sin \alpha + \sin 2\alpha + \dots + \sin (n - 1)\alpha)). \end{aligned}$$

This formula can be derived from Eq. (1) (or indeed directly from *prosthaphaerisis*) and is easier to derive than Roegel’s formula above. It was, in fact, explicitly derived in the *Yuktibhāṣa* (Sarma et al. 224f.) in the same context of calculating sine tables. With the approximation $R \sin j\alpha \approx jR \sin \alpha$, which Būrgi uses for small angles (e.g., the approximation $\sin 1' \approx \frac{\sin 1^\circ}{60}$, where $j = 60$ and $\alpha = 1'$), the above formula simplifies to:

$$\begin{aligned} R \sin 1' &\approx R \frac{\sin 1^\circ}{60} + \frac{1}{60} \left(\frac{Rcrd1'}{R} \right)^2 R \sin 1' (1 + (1 + 2) + \dots + (1 + 2 + \dots + 59)) \\ &= R \frac{\sin 1^\circ}{60} + \Delta_{1,1'}^2 \frac{1}{60} \left(\frac{59 \cdot 60 \cdot 61}{6} \right). \end{aligned}$$

The same approximation is also used in the *Yuktibhāṣa* (Sarma et al. 2009, 226, 229).

This reconstruction provides us with the same coefficient for the second difference as Būrgi’s, without having to go through third-order differences. It does not, however, explain why Būrgi would present $\frac{1}{60} \left(\frac{59 \cdot 60 \cdot 61}{6} \right)$ as the sum $\frac{30 \cdot 60}{4} + 2 \frac{(1+2+\dots+29) \cdot \frac{31}{3}}{60}$. One possibility is that this sum of two terms simply results from the specific formula that Būrgi used to calculate the sum $1 + (1 + 2) + \dots + (1 + 2 + \dots + 59)$. However, we could not come up with an analysis that would “naturally” lead to such a formula. Another possibility is that Būrgi wanted to obscure his procedure and therefore used his two-term sum instead of the simpler single term. We know that Būrgi tried to keep his methods secret (see Lauenert 2015, comments on ch. 10; Staudacher 2018, ch. 12), a practice that was

⁴ I am assuming here, following Roegel (2016), that a division of the second term by 60 was indeed intended by Būrgi, and that the first term is indeed multiplied by 60. Roegel shows that the numbers do not make sense otherwise. The fact that Būrgi was inconsistent in his notation of place values in sexagesimal numbers (e.g., Lauenert 2015, 18) lends plausibility to this reconstruction.

indeed common at the time among craftsmen and scientists alike. (Tartaglia is a famous example from the history of mathematics; for a general survey, see Davids 2005.)

Yet another possibility is that Bürgi wanted to calculate with smaller numbers and therefore broke the single term formula $\frac{59 \cdot 60 \cdot 61}{6}$ into two smaller terms. This decomposition can be derived from dissecting and recomposing the triangular pyramid that corresponds to the sum $1 + (1 + 2) + \dots + (1 + 2 + \dots + 59)$ into a cube with side 30 and two smaller pyramids, each corresponding to $1 + (1 + 2) + \dots + (1 + 2 + \dots + 29)$.

A further point is Bürgi's instruction to calculate the sum of squares of the numbers from 1 to 29, which is not actually used. Launert (2015, 66) suggests that this would emerge from constructing a sine table with resolution 3° starting from a linear approximation of sines using a single iteration of the *Artificium*. Such a table, however, is not mentioned by Bürgi and, even if it were, is irrelevant for the context of calculating the sine of a single arc minute.

I find it more likely that Bürgi's mention of the sum of squares refers to a decomposition of the sum $\sum_i \left(\sum_{j=1}^i j \right)$ into terms that involve $\sum i^2$. These terms may have led Bürgi to his final two-term formula, but in such a way that $\sum i^2$ canceled out or transformed into something else and therefore disappeared from the final formula. One could also interpret the mention of the sum of squares as referring to *another* method of calculation (reading "Zum andernn" as "alternatively"), which is not elaborated in the text. Such alternatives are indeed offered in the *Yuktibhāṣa* (Sarma et al. 2009, 227f., 230).

3.4 The calculation of the sine of further minutes

Launert explains Bürgi's procedure for calculating the sine of subsequent minutes as derived from the formula $R\Delta_{j,1}^2 \approx j \cdot R\Delta_{1,1}^2$, which follows from (1) and $R \sin j\alpha \approx jR \sin \alpha$. Successively adding the second differences yields first differences, and adding these yields in turn the actual sines.

Almost the same approach was suggested by Nīlakaṇṭha for constructing sine tables and, according to Hayashi (1997), was used even by Āryabhāṭa almost a millennium earlier. Note, however, that Nīlakaṇṭha uses the exact formula $R\Delta_{j,\alpha}^2 = \frac{R \sin j\alpha}{R \sin \alpha} \cdot R\Delta_{1,\alpha}^2$ rather than the simplified approximation above. Given the R sines up to $j\alpha$, the latter formula allows to derive $R\Delta_{j,\alpha}^2$, which (again, given the R sines up to $j\alpha$) allows to derive $R \sin (j + 1)\alpha$, and so on.

Bürgi could rely on the approximation $R\Delta_{j,1}^2 \approx j \cdot R\Delta_{1,1}^2$, because he was applying it to small angles, where his approximation makes sense. Nīlakaṇṭha, on the other hand, applied it to larger angles, where the approximate formula would yield poor results. To bridge the gap, Bürgi notes at the very end of his instructions that one may want to update (possibly by means of *prosthaphaeresis*) the value of the difference of difference as the angles increase.

3.5 How likely is the case for transmission?

The transmission of astronomical and mathematical knowledge from India to Europe was suggested as an explanation for the emergence of the new astronomy and calculus of early modernity (e.g., Bala 2006, ch. 7; Raju 2007; Joseph 2009a; b, ch. 9). However, while there is good evidence for motivation and opportunity for the transmission of knowledge, hard evidence for actual transmission in the form of relevant Indian manuscripts, translations, or obviously borrowed pieces of knowledge is still missing.

Bürge's case does not change this reality. We do not present here conclusive evidence of transmission. The evidence depends on analogies between Bürge's work and that of the Kerala School, and there are reasonable alternative narratives that would account for an independent European discovery. However, we believe that Bürge's case is a stronger *candidate* for transmission than other cases reviewed in the literature. Indeed, one argument against transmission is that the early infinitesimalist mathematical developments in Europe relate to a different kind of questions than those considered by the Kerala school (Plofker 2009, 252f.). In Bürge's case, however, the European and Indian contexts are the same: constructing sine tables.

As we noted above, we find it difficult to argue that Bürge's iterative construction (the *Artificium* itself, namely the columns of values generated by partial sums) had an Indian origin. Bürge's method can be viewed as an elaborate application of a fixed-point iteration to an array of values, which, given the evidence available to us, appears to be original and unique even against the Indian and Arabic background. So even if Bürge's knowledge depended on Indian sources, it is still likely that the *Artificium* itself was his own innovation. What may have been transferred is some version of the identity (1) and the procedures involved in calculating the *R*sines of minutes by means of a second difference.

This leaves the question of how the knowledge was transferred. We do not suggest a written transmission of a full major treatise, as this would have had to involve European scholars cooperating with Sanskrit or Malayalam readers on a written translation. In view of the intellectual effort required for such a task and the striking innovations that it would have presented, such a translation would most likely have made much more impact than Bürge's arcane work.

Oral transmission, however, is more likely. The networks that connected India and Europe must have included people interested in sine calculations for the purposes astral sciences, calendrics, and navigation. Such people, who may have been lost to historical record, could deliver fragments of knowledge. Indeed, Scaliger (1583, 231f.) and Kepler (Joseph 2009b, 262) relied on Indian astronomical or calendrical knowledge, so the contacts were there (see also Bala 2006, ch. 7; Raju 2007; Joseph 2009a, ch. 9).⁵ Some relevant mathematical components may have reached Bürge through Wittich (who introduced him to *prosthaphaeresis*, according to Gingerich

⁵ Juan Marín is currently collecting additional evidence for such transmission, but it is too early to claim that any of this is relevant for Bürge's case.

and Westman 1988, 17, 68), or through Ursus, John Dee, or other travelers,⁶ or perhaps even during the travels of Bürgi's youth. This would provide Bürgi with the components for his later innovations.

In fact, already in 1505 the German mercantile patriciate, including the Fugger and Welser family trading houses, invested in the Portuguese trading missions to southwest India (Lach 1994, 108f). Many of these trading houses had their headquarters in or around either Augsburg or Nurnberg, from where they maintained a vast network of factories. According to Staudacher, it is possible that Bürgi was in both these cities during his journeyman years (2018, 60ff.), long after Indo-Germanic trade started. Moreover, in addition to his interest in astronomy, Landgrave William IV of Hesse-Kassel collected "exotic" animals and plants (Werner 2013, 29ff.). He built one of the first botanical gardens in Germany (Hanschke 1991) and was in possession of a herbarium that contained plants from India, Turkey, and America (Jaeger 2007, 379). Hence, Bürgi lived in the proximity of people directly or indirectly connected to India and might have come into contact with some of them in Augsburg, Nurnberg, or Kassel.

Note that this transmission theory must either assume that all public records of the transmission were lost or explain why none of the European agents in the chain of transmission from India to Bürgi publicized the results or claimed them for themselves. The reasons may be varied: failure to fully understand or appreciate the mathematical content, lack of trust in unproven and hard to verify results, difficulty in usefully applying the transmitted knowledge, or a social position that hindered access to the means of dissemination of knowledge. We should note that some of these motivations may have applied to Bürgi himself, who did not really publicize the results. He made a note of them in a single manuscript presented to Emperor Rudolf II (Launert 2015) and included a cryptic reference in another manuscript (List and Bialas 1973). In the former, his formulations are mangled in the most critical points, possibly deliberately so, in order to guarantee a monopoly over his knowledge. Perhaps other agents along the chain acted similarly, but left us with no tangible traces.

All this is indeed highly speculative, so this paper does not pretend to contain a conclusive proof of transmission. However, it does somewhat strengthen the likelihood of transmission and might help us narrow down the search.

Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

⁶ Staudacher (2018, 221ff.) suggests that Dee transferred Bürgi's knowledge to England, rather than the other way around, but there is no strong evidence either way.

Appendix: Translation of “The eleventh chapter concerning the production of the Sine–Canon by the division of a right angle in as many parts as one wishes”

This translation is based on Launert’s edition (2015). We attempted to strike a reasonable compromise between literal translation and intelligibility. Our interpellations are added in square brackets, and clarifications or corrections in curly brackets.

The most desired principle
that the eager meditations of Bürgi reveal
as it has never been seen or existed before.

Divide a right angle in as many parts as desired, and produce from the latter the Sine–Canon. First, place several numbers of your choice over each other in a column, as many as the parts in which you want to divide the given right angle, and take half of the last number among those placed numbers, and place this half against that last number on the left, but elevated by about half a number, so that it stands between this last one and the next number above it in the form of a difference number.⁷ To this half add or join the last but one among the initially chosen numbers, and in turn to the arising result the adjacent one before it among the chosen numbers, and again to this arising result the adjacent-preceding among the chosen numbers, and so forth until the first chosen or initially placed number. Thereupon, in this manner arise the differences of the given parts of the angle or the differences between the content of its parts {that is, the content of the corresponding cells}. Therefore, sum together these differences in turn from top to bottom, each with the next below it. Thus, arise to some extent the desired parts of the given angle, but not so precisely.

Operate therefore further with these found parts as with the initially placed and chosen numbers before, and the longer and more you practice thus with these parts and their differences, each with respect to the other, and proceed and advance to the left, the more precise and certain arises from this the content of the parts of the angle, until they finally no longer or further change in their proportions, and remain almost constant, which will give the one proper and certain sign that this aforementioned right angle is divided as desired. An example in logistic or astro-nomic {that is, sexagesimal} numbers follows, in which the right angle is divided in nine parts.

⁷ This apparently relates to a practice of aligning the terms of a difference sequence of a given sequence with the spaces between the terms of the given sequence.

Example: (Bürgi 1592, 36r)

	<i>sinūs</i> 5	<i>sinūs</i> 4	<i>sinūs</i> 3	<i>sinūs</i> 2	<i>sinūs</i> 1
0	0. 0. 0. 0. 0	0. 0. 0. 0. 0	0. 0. 0. 0. 0	0. 0. 0. 0. 0	0
10	10. 20. 51. 0	0. 18. 51. 52	0. 34. 24	1. 3	2
20	20. 22. 50. 3	0. 37. 9. 20	1. 7. 45	2. 4	4
30	29. 47. 39. 56	0. 54. 19. 3	1. 39. 2	3. 1	6
40	38. 18. 10. 41	1. 9. 49. 44	2. 7. 18	3. 52	7
50	45. 38. 51. 42	1. 23. 13. 7	2. 31. 42	4. 36	8
60	51. 35. 19. 35	1. 34. 4. 48	2. 51. 30	5. 12	9
70	55. 59. 42. 42	1. 42. 4. 59	3. 6. 6	5. 39	10
80	58. 41. 0. 49	1. 46. 59. 4	3. 15. 3	5. 56	11
90	59. 35. 19. 52	1. 48. 38. 6	3. 18. 4	6. 2	12

This example also proceeds a bit further and therefore consists of more precise sines, so that those being trained may proceed somewhat more proficiently (Bürgi 1592, 36r).

<i>gradus</i>	<i>sinūs</i>	<i>sinūs</i>	<i>sinūs</i>	<i>sinūs</i>	<i>sinūs</i>	<i>sinūs</i>	<i>sinūs</i>
0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0	0. 0. 0. 0. 0. 0. 0
10	1. 42. 27. 54. 56. 49. 39	3. 5. 48. 4. 43. 2	5. 40. 33. 6. 30	10. 20. 51. 0	18. 51. 52	34. 24	1. 3. 2
20	1. 32. 13. 11. 15. 47. 3	2. 49. 55. 45. 14. 32	5. 9. 49. 29. 22	9. 24. 49. 48	17. 9. 43	21. 17	5. 7. 2
30	4. 55. 2. 13. 7. 42. 19	8. 57. 52. 22. 34. 6	16. 20. 34. 47. 22	29. 47. 39. 66	54. 19. 3	1. 39. 2	3. 1. 6
40	1. 24. 15. 18. 53. 12. 57	2. 33. 35. 11. 27. 10	4. 40. 1. 45. 26	8. 30. 30. 45	15. 30. 41	28. 16	5. 1. 1
50	6. 19. 17. 31. 57. 56. 16	11. 31. 28. 37. 1	15. 21. 0. 36. 32. 48	38. 18. 10. 41	1. 3. 49. 47	2. 7. 18	3. 5. 2. 7
60	1. 12. 43. 50. 19. 11. 41	2. 12. 35. 34. 54. 22	4. 1. 41. 34. 45	7. 20. 41. 1	13. 27. 33	24. 24	44. 1
70	7. 32. 1. 22. 17. 7. 57. 13. 44	4. 8. 55. 38	25. 22. 0. 7. 33. 45. 38. 51. 42	1. 23. 17. 7	2. 31. 42	4. 35. 8	
80	8. 31. 1. 8. 27. 24. 0	15. 31. 37. 23. 42. 27	28. 18. 24. 50. 36	51. 35. 19. 36	1. 34. 4. 48	2. 51. 30	5. 12. 9
90	9. 41. 0. 33. 46. 52. 32	17. 39. 24. 10. 20. 50	32. 11. 21. 54. 48	58. 41. 0. 49	1. 46. 59. 4	3. 15. 3	5. 56. 11

Several tests and verifications follow.

Because this construction is grounded in arithmetic and not in geometry, it is not possible to visually {that is, geometrically} demonstrate it, but it has to be satisfactorily tested in several ways as follows.

1. The sine of 30 degrees is half the radius.
2. The square of the radius is equal to the square of a sine of an arc and the sine of the complement of that arc.

3. The sines of the arcs having the same difference under and over 60, the smaller subtracted from the larger, leave the sine of this difference counted in degrees and minutes, of which will be said more in the next chapter, including its demonstration.
4. All kinds of different sines {that is, sine approximations in different columns} of the same degrees stand to each other approximately in one proportion, as well as the differences [of sines] of the same degrees, the larger [the values], the more precisely.
5. Ultimately one may try to test it also by means of the inscription,⁸ the sines and their proportion. From all those trials together and from each of them separately, the certainty of these things will appear satisfactorily and clearly. So much briefly about the division of the angle. Now follows how one produces the Canon from this [construction].

Divide the right angle according to the presently given teaching, just as into nine, so also similarly into ninety parts. Thereupon, you will obtain the proportion or content of the sines of ninety degrees in the Canon or quadrant [of a circle], respectively, but with an inconvenient and not satisfactorily adequate radius (namely that which arises from the effected addition). For it is not under our control or within our choice to get or obtain this radius (however or in whatever numbers one wants). But from that found content one may now easily, by means of the rule of proportion or of three, reduce and bring the arising radius or the last number attained to a much more adequate number and to any radius. Namely like this: as the arising radius is to the desired radius, thus the other arising sine (compare same degrees to the same degrees) is to the sine of the desired radius in their content, and in this manner one may reduce {that is, normalize} the sine or this Canon, drawing on whatever adequate number as one wishes, and thereupon, the sines stand in the same proportion to their desired radius as the sines, which initially emerge from the division of the angle, to their emergent radius or largest sine.

And in the same manner, one may obtain the sines of all minutes between any two degrees, namely by means of the division of the right angle into 5400 parts (because so many minutes are in the Canon or quadrant). But because the work of the presently mentioned reduction through so many proportions of minutes, and moreover, the ascending and descending addition, might turn out quite difficult, one may seek the sines of all minutes between any two degrees much more easily and proficiently from the sines of the degrees, their differences and their adjustment to the unequal increase of the sines in the circle {that is, to the fact that sines do not increase linearly with the angle}. But the origin and ground of such an inquiry lies in the search for the true sine of the first minute in the Canon, which may be found in full precision quickly and proficiently through the following process. Namely like this.

⁸ This refers to the calculation of sines by means of geometric analysis of inscribed polygons, as Bürgi does earlier in the treatise.

- I. First, from the completed division of the angle into 90 parts, the sine of all degrees is known as well as the sine of the first degree, from which, when divided into sixty parts (namely the number of minutes of a degree), the sine of the first minute in the Canon emerges approximately. But because of the unequal increase of the sines in the circle, it is somewhat too small, but nonetheless sufficient for the following.
- II. Now continuing, the square of this sine of the first minute subtracted from the square of the radius leaves the square [of the sine of the right angle] without the last minute, albeit like the first one {that is, like the sine of the first minute}, only approximately. Then, similarly to the sine of the first minute, it comes out somewhat too big, but nonetheless here it is sufficient.
- III. From such approximately known sine [of the right angle] without a last minute, that is, with respect to the radius, the difference between those two sines {that is, the difference between the radius and the sine of a right angle less one minute} will also be immediately known, of which the half {we suggest "half" should be replaced by "double"} will be (in the manner of the given teaching concerning the division of the angle) the radius or very last and biggest sine of the other previous kind or distinct sines.⁹
- IV. Now, as the radius of the right or first kind of sines is to the presently mentioned [radius] of the other, namely to the radius of the preceding kind of sines, so is the sine of the first degree of the first kind of sines to the sine of the first degree of the other or preceding kind of sines, from which its sixtieth is found. {We suggest adding: This proportion multiplied by the sine of the first minute will}¹⁰ be the difference of differences between the sine of the first and next {that is, second} minutes in the Canon. From, which hereafter the proper and true size of the sine of the first minute will be further investigated and found through two different added terms, as follows:

⁹ Launert suggests another reading and interpretation. First, he reconstructs Bürgi's missing right hand parentheses differently than we do. Moreover, his reading hypothesizes a missing "square of the" before "previous kind or distinct sine," as he reconstructs this step as expressing the formula $\sin^2 \frac{\alpha}{2} = \frac{1}{2}(1 - \cos \alpha)$. Launert's reading also requires interpreting the "previous kind or distinct sine" inconsistently with the use of this phrase in the next step and poses some grammatical difficulties. Our reading interprets this step as saying that the difference between the two bottom terms of an odd column is the bottom term of the even column to its right, which is half the bottom term in the preceding odd column (the "previous kind or distinct sine"). This reading also requires a manipulation: replacing the "half" by "double," which may be due to Bürgi accidentally looking at the relevant identity from the wrong direction. We believe that our reading makes more sense for the given context.

¹⁰ The text is grammatically defective here, as the infinitive *sein* depends on no finite verb, and the content does not fit an independent infinitive clause. If this error is due to a line omitted in copying, our context-based reconstruction is indeed plausible. (Launert 2015, 59): "Wie nun der radius der rechten oder ersten art Sinuum gegen denselben Itzgemelten der andernn, Nemblich der vorhergehendenn ardt Sinuum Iren radium, also der Sinus deß erstenn grads deß Ersten art Sinuum, gegen den Sinum des ersten grads der andernn oder vorhergehendenn art Sinuum, auß welcher gefundenn wirt ein sechzigtheil deßelben, sein die Differenz der Differenzen zwischen den Sinibus der erstenn vnnd andern minuten im Canone".

- V. To find the first added term: multiply the presently found difference of differences by 30, a quarter of the arising¹¹ product will be the first added term.
- VI. To find the next added term: Seek the sum of the familiar natural number progression from one to 29, because of the 29 differences between 1 and 30 min; 7' 15" comes out.¹² Next {or: alternatively} seek the sum of the progression of the square numbers as well. Add the first number of the progression to its end, that is, 1 to 30, becomes 31, divide this by 3, 10' 20" comes out. Multiply by that the previous 7' 15", from which arises 1' 14" 55"". Multiply by that again the mentioned difference of differences—doubling the product that comes out will be the other added term, which needs to be added to the first added term [and to] the initial, approximate, previously calculated sine of the first minute, namely the sixtieth part of the sine of one degree. From these three then arises the correct and true sine of the first minute in the Canon, from which the sines of all subsequent minutes may now be further sought easily with full precision, and also from degree to degree or between any two neighboring degrees, only by means of prosthaphaeresis and without any further application of proportion. Namely like this: The found sine of the first minute is at the same time also the difference between the sines of the same first- and of no-minute, from which difference the found and accepted difference of differences is subtracted once; there remains the difference between the sines of the first and next {that is, second} minutes. From, which again the mentioned difference of differences is subtracted twice; remains the difference between the sines of the next {that is, second} and the third minute. And so on always, one subtracts three, four, five, six, seven, etc., times the difference of differences from the difference of the sines, according to the order and succession of the number of minutes. There always remains the proper difference between the sines of each pair of neighboring minutes. Therefore, whenever the remaining difference [between two consecutive sines] is added to the sine of some minute, the sine of the following minute arises, and so on until the end of the Canon. Nevertheless, in all the following degrees, the differences of differences may be changed twice or more due to the unequal increase of the sines in the quadrant according to the nature of the differences of the sines between any two degrees. Therefore, the differences of differences stand as the number and increase of a natural progression {that is, they grow proportionally to the sequence 1, 2, 3...}, from which they may also be easily tested at the end of all degrees or half degrees, and so all that is unknown is perfected and produced in our crafted canon.

¹¹ It seems that Roegel (2016) reads this word (“erwachßen”) as signifying an upward shift of one sexagesimal place. We accept his quantitative reconstruction, but do not think that this word testifies to it explicitly. According to Roegel, the downward shift that his reconstruction requires for the next added term was erroneously left out.

¹² Note that here and below, the most significant digit of all sexagesimal numbers is marked as minutes, rather than its correct place value (sixties, units, degrees, minutes, etc.). This makes the reconstruction of the calculation difficult. In our commentary, we follow Roegel’s (2016) interpretation.

References

- Bala, Arun. 2006. *The dialogue of civilizations in the birth of modern science*. New York: Palgrave Macmillan.
- Bürgi, Jost. 1592. "Fundamentum Astronomiae." Manuscript no. IV Q 38 a. Biblioteka Uniwersytecka in Wrocław. <http://www.bibliotekacyfrowa.pl/dlibra/publication/43578/edition/45624>, Accessed July 3 2019.
- Clark, Kathleen M. 2015. *Jost Bürgi's Arithmetische Und Geometrische Progreß Tabulen (1620)*. Basel: Birkhäuser.
- Davids, Karel. 2005. Craft secrecy in Europe in the early modern period: A comparative view. *Early Science and Medicine* 10 (3): 341–348.
- Divakaran, P.P. 2018. *The Mathematics of India: Concepts, Methods, Connections*. Singapore: Springer.
- Folkerts, Menso, Dieter Launert, and Andreas Thom. 2016. Jost Bürgi's method for calculating sines. *Historia Mathematica* 43 (2): 133–147. <https://doi.org/10.1016/j.hm.2016.03.001>.
- Gaulke, Karsten (ed.). 2007. *Der Ptolemäus von Kassel. Landgraf Wilhelm IV. von Hessen-Kassel und die Astronomie*. Kassel: Museumslandschaft Hessen-Kassel.
- Gingerich, Owen, and Robert S. Westman. 1988. The Wittich connection: Conflict and priority in late sixteenth-century cosmology. *Transactions of the American Philosophical Society* 78 (7): i–i148. <https://doi.org/10.2307/1006552>.
- Hanschke, Ulrike. 1991. Die Gartenanlagen der Landgrafen Wilhelm IV. und Moritz in Kassel im Spiegel handschriftlicher Quellen. *Die Gartenkunst* 3 (2): 175–188.
- Hayashi, Takao. 1997. Aryabhata's rule and table for sine-differences. *Historia Mathematica* 24 (4): 396–406. <https://doi.org/10.1006/hmat.1997.2160>.
- Jaeger, Friedrich (ed.). 2007. *Enzyklopädie der Neuzeit*, vol. 5. Stuttgart: J.B. Metzler.
- Joseph, George Gheverghese. 2009a. *A passage to infinity: Medieval Indian mathematics from Kerala and its impact*. New Delhi: Sage Publications.
- Joseph, George Gheverghese (ed.). 2009b. *Kerala mathematics: History and its possible transmission to Europe*. Delhi: B.R. Publishing Corporation.
- Lach, Donald F. 1994. *Asia in the making of Europe, Book 1*, vol. 1. Chicago: University of Chicago Press.
- Launert, Dieter. 2015. *Bürgis Kunstweg im Fundamentum Astronomiae: Entschlüsselung seines Rätsels*. München: Bayerische Akademie der Wissenschaften München. <http://publikationen.badw.de/de/043287045>. Accessed July 3 2019.
- List, Marta, and Volker Bialas. 1973. *Die Coss von Jost Bürgi in der Redaktion von Johannes Kepler: Ein Beitrag zur frühen Algebra*. München: Bayerische Akademie der Wissenschaften München. <http://publikationen.badw.de/de/003090688>. Accessed July 3 2019.
- Nicollier, Grégoire. 2018. How Bürgi computed the sines of all integer angles simultaneously in 1586. *Mathematische Semesterberichte* 65 (1): 15–34. <https://doi.org/10.1007/s00591-017-0209-0>.
- Plofker, Kim. 2009. *Mathematics in India*. Princeton: Princeton University Press.
- Plofker, Kim. 2011. Why did sanskrit mathematics ignore Asakrt methods? In *Scientific literature in Sanskrit*, vol. 1, ed. S.R. Sarma and G. Wojtilla, 61–76. Delhi: Motilal Banarsidass.
- Raju, C.K. 2007. *Cultural foundations of mathematics: The nature of mathematical proof and the transmission of the calculus from India to Europe in the 16th c. CE.* Delhi: Pearson Longman.
- Ramasubramanian, K., and M.S. Sriram. 2010. *Tantrasangraha of Nilakantha Somayaji*. London: Springer.
- Roegel, Denis. 2015. *Jost Bürgi's skillful computation of sines*. hal-01220160. <https://hal.inria.fr/hal-01220160/document>. Accessed July 3 2019.
- Roegel, Denis. 2016. *A preliminary note on Bürgi's computation of the sine of the first minute*. hal-01316358. <https://hal.inria.fr/hal-01316358/document>. Accessed July 3 2019.
- Sarma, K.V., K. Ramasubramanian, M.D. Srinivas, and M.S. Sriram. 2009. *Ganita-Yukti-Bhasa of Jyesthadeva*. Berlin: Springer.
- Scaliger, Joseph Justus. 1583. *Opus Novum de Emendatione Temporum*. Basel: Roberti Stephani.
- Šíma, Zdislav. 1993. Prague sextants of Tycho Brahe. *Annals of Science* 50: 445–453. <https://doi.org/10.1080/00033799300200331>.
- Staudacher, Fritz. 2018. *Jost Bürgi, Kepler und der Kaiser: Uhrmacher, Instrumentenbauer, Astronom, Mathematiker, Erz-Metallurgist*, 4th ed. Zürich: NZZ Libro.

- Ullrich, Peter. Forthcoming. An educated guess how Jost Bürgi may have come to his 'Artificium' for the calculation of values of the sine function.
- Ullrich, Peter. 2016. The mathematics behind Jost Bürgi's method for calculating sine tables. *Proceedings in Applied Mathematics and Mechanics* 16 (1): 891–892. <https://doi.org/10.1002/pamm.201610434>.
- von Mackensen, Ludolf. 1979. *Die erste Sternwarte Europas mit ihren Instrumenten und Uhren: 400 Jahre Jost Bürgi in Kassel*. Munich: Callwey Verlag.
- Wagner, Roy. 2015. Citrabhānu's twenty-one algebraic problems in Malayalam and Sanskrit. *Historia Mathematica* 42 (3): 263–279. <https://doi.org/10.1016/j.hm.2015.01.001>.
- Waldvogel, Jörg. 2016. Jost Bürgi's Artificium of 1586 in modern view, an ingenious algorithm for calculating tables of the sine function. *Elemente der Mathematik* 71 (3): 89–99. <https://doi.org/10.4171/EM/304>.
- Werner, Petra. 2013. *Die Menagerie des Landgrafen Karl: Ein Beitrag zur Einheit von Natur und Kunst im Barockzeitalter*. Ph.D. dissertation, University of Kassel.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.